THE PHYSICAL BASIS OF MAGNETOHYDRODYNAMICS

By W. B. Thompson

Atomic Energy Research Establishment, Harwell

1. INTRODUCTION

MAGNETOHYDRODYNAMICS is the study of the flow of electrically conducting fluids interacting with the electromagnetic field. It is a subject which has been developed quite recently, first probably by astrophysicists and geophysicists interested in sunspots and the aurorae (e.g. Chapman and Ferraro, Cowling), the earliest laboratory experiments, since Faraday, probably being those of Hartmann on the flow of mercury, experiments which have led to the development of magnetic pumping of liquid metals. In the past few years the hope of obtaining power from controlled thermonuclear reactions in a magnetically confined plasma (ionized gas) has given great stimulus to the subject, and quite recently the possibility of using fluid field interactions in systems of ionic drive, magnetohydrodynamic braking, and magnetic insulation has begun to intrigue the aeronautical scientist.

The interaction between fluid and field occurs first through the body force between the current density $\mathbf{j}$ and the magnetic field $\mathbf{B}$

$$\mathbf{F} = \mathbf{j} \times \mathbf{B}$$

and secondly through the effect of the induced current $\mathbf{j}$ on the magnetic field, given by Maxwell's relation

$$\text{curl } \mathbf{B} = 4\pi \mathbf{j}.$$

These processes are necessarily complex, but in certain idealizations can be represented by the concepts of "the magnetic pressure" and "the frozen-in field". I shall be concerned with basic physical processes which underlie the behaviour described by these expressions, particularly in extremely hot ionized gases, probably the most interesting case.

2. MATHEMATICAL DESCRIPTION OF MAGNETOHYDRODYNAMICS

In a subject such as this we must depend heavily on mathematical

methods; and as one starting point we can use the Navier–Stokes equation coupled to Maxwell's equations for the field.

$$\rho \frac{D\mathbf{u}}{Dt} = - \nabla p - \nu(\text{curl curl } \mathbf{u}) + \frac{4}{3} \nu \text{ grad div } \mathbf{u} + \mathbf{j} \times \mathbf{B} + q \cdot \mathbf{E} \quad (2.1)$$

where ν is the viscosity and

$$\text{curl } \mathbf{H} = 4\pi\mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{D}}{\partial t} \quad (2.2)$$

$$\text{curl } \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}. \quad (2.3)$$

$$\text{div } \mathbf{B} = 0, \text{ div } \mathbf{D} = 4\pi q.$$

In addition we will need constitutive equations, $\mathbf{H}=(1/\mu)\mathbf{B}$ and $\mathbf{D}=K\mathbf{E}$, the equation of continuity, some equation determining p, and one determining $\mathbf{j}$. The flow is often assumed adiabatic ($\nu=0$) $p \sim \rho^\gamma$, and the current determined by a local Ohm's law,

$$\mathbf{j} = \sigma\mathbf{E}^* = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \quad (2.4)$$

where σ is the electrical conductivity, and $\mathbf{E}^*$, the local electric field, i.e. the electrical force acting on a unit charge moving with the local velocity $\mathbf{u}$ of the fluid. The magnetohydrodynamic terms are made more transparent on being transformed. If $\mathbf{j}$ in (2.1) is replaced by (2.2), with the displacement current $(1/c^2)(\partial\mathbf{D}/\partial t)$ neglected and μ assumed constant, we obtain

$$\mathbf{j} \times \mathbf{B} = \frac{1}{4\pi\mu} (\text{curl } \mathbf{B}) \times \mathbf{B} = - \frac{1}{8\pi\mu} [\overline{\text{grad}} \, B^2 - 2(\mathbf{B} \cdot \nabla) \mathbf{B}] \quad (2.5)$$

which represents a non-isotropic pressure $B^2/8\pi\mu$ acting at right angles to the field lines. For many problems the charge density q vanishes, and this magnetic pressure is the sole dynamical effect of electromagnetic field. If we operate with *curl* on (2.2) and use the generalized Ohm's Law (2.4) to eliminate $\mathbf{j}$ (with σ a constant), then use (2.3) to eliminate curl $\mathbf{E}$ we obtain

$$\frac{1}{4\pi\mu\sigma} \text{curl} (\overline{\text{curl}} \, \mathbf{B}) = [\overline{\text{curl}} \, \mathbf{E} + \overline{\text{curl}} (\mathbf{u} \times \mathbf{B})]$$

$$= - \frac{\partial \mathbf{B}}{\partial t} - (\mathbf{u}.\nabla)\mathbf{B} + (\mathbf{B}.\nabla)\mathbf{u} - \mathbf{B} \text{ div } \mathbf{u}$$

or

$$\frac{D\mathbf{B}}{Dt} + \mathbf{B} \text{ div } \mathbf{u} - (\mathbf{B}.\nabla)\mathbf{u} = \frac{1}{4\pi\mu\sigma} \nabla^2\mathbf{B}. \quad (2.6)$$

The terms on the left of this equation represent a convection of field lines with the fluid motion, while that on the right represents the diffusive penetration of field into a conducting medium.

3. MAGNETOHYDRODYNAMIC REGIONS

Examination of these equations discloses the conditions for typical

magnetohydrodynamic behaviour. From (2.1) and (2.5) it is clear that, if the magnetic field and the motion are to interact strongly, the kinetic energy of motion $\frac{1}{2}\rho u^2 \simeq B^2/8\pi\mu$, or

$$\frac{B^2}{4\pi\mu\rho u^2} \simeq 1. \tag{3.1}$$

For equality $u = c_A = \sqrt{B^2/4\pi\mu\rho}$, the Alfvén velocity. Also, if the convection of field lines is to be more important than diffusion, i.e. if the flow is seriously to affect the field, we must have

$$(\mathbf{u}.\nabla)\mathbf{B} > \frac{1}{4\pi\mu\sigma}\nabla^2\mathbf{B}$$

i.e.

$$uL4\pi\mu\sigma > 1 \tag{3.2}$$

where L is a characteristic distance for variations in the magnetic field. These two conditions may be combined to give the condition for "typical" magnetohydrodynamic behaviour.

$$M = BL\sigma\sqrt{\frac{4\pi\mu}{\rho}} >> 1. \tag{3.3}$$

The first table gives the value of M under simple laboratory conditions, i.e. $L = 10$ cm, $B \simeq 1000$ g for various fluids. ($\mu \simeq 1$ throughout.)

Conditions for "typical" magnetohydrodynamic behaviour on laboratory scale $B = 1000$ g, $L = 10$ cm, $\mu = 1$ $M >> 1$

Fluid	Density ρ g cm^{-3}	Conductivity σ e.m.u.	$M = BL\sigma\sqrt{\frac{4\pi\mu}{\rho}}$
Hg	13·5	10^{-5}	0·1
Na (liquid)	0·93	10^{-4}	4·0
5% ionized gas $n_i = 10^{16}$ $T = 10^4$	10^{-8}	$2\cdot3 \times 10^{-6}$	10^3
Hot plasma $n = 10^{15}$ $T = 10^6$	10^{-9}	3×10^{-5}	3×10^4
Thermonuclear plasma $n = 10^{16}$ $T = 10^8$	10^{-8}	3×10^{-2}	10^7

4. IDEALIZED MAGNETOHYDRODYNAMICS

If the magnetic pressure $B^2/8\pi$ is comparable to the hydrostatic pressure p the field can be expected to affect slow motions of the fluid, and since a field of $\sim$5000 g has an associated magnetic pressure of $\sim$1 atmosphere, dynamic effects are easily produced by strong magnetic fields. However, if the conductivity is low the diffusive term in (2.6) is dominant, the magnetic effects are dissipative and motions are magnetically damped, unless the currents are maintained from external sources as in a magnetic pump. However, if the conductivity, or more properly $4\pi\mu\sigma Lu$, is large, dissipative processes are unimportant and, for studies of its motion, the fluid may be

considered as a perfect electric conductor. In this case we replace Ohm's law by the statement that the local electric field vanishes,

$$\mathbf{E}^* = \mathbf{E} + \mathbf{u} \times \mathbf{B} = 0. \tag{4.1}$$

The development of the field is then determined by the appropriate form of (2.6)

$$\frac{D}{Dt}\mathbf{B} + \mathbf{B} \operatorname{div} \mathbf{u} - (\mathbf{B}.\nabla)\mathbf{u} = 0. \tag{4.2}$$

This should be compared with the equation governing the vorticity ψ of a non-viscous fluid

$$\frac{D\psi}{Dt} + \psi \operatorname{div} \mathbf{u} - (\psi . \nabla)\mathbf{u} = 0,$$

hence, by analogy, Kelvin's theorem holds for magnetic field, i.e. the flux encircled by a closed curve moving with the fluid is constant. This reflects the vanishing of the electric field in the fluid, for Maxwell's equation (2.3) indicates that the rate of change of flux threading a loop is proportional to the electromotive force around the loop, i.e.

$$\frac{D\varphi}{Dt} = \frac{D}{Dt}\int \mathbf{B}.d\mathbf{S} = \oint \mathbf{E}^*.d\mathbf{l}$$

but since $\mathbf{E}^* = 0$, $D\varphi/Dt = 0$.

This trapping of magnetic flux leads to the concept of the "frozen-in field", the field lines being frozen to the fluid.

If the field lines are straight and the flow occurs in planes normal to them, the magnetohydrodynamic relations become particularly simple,

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla\left(p + \frac{B^2}{8\pi}\right), \frac{D}{Dt} B/\rho = 0; \tag{4.3}$$

which shows that the fluid moves as though mixed with an isentropic inertialess component with $\gamma = 2$; $p_T = p + p_H$, $p_H = (B^2/8\pi) \sim \rho^2$.

5. TYPICAL MAGNETOHYDRODYNAMIC PHENOMENA

The simple concepts of magnetic pressure and the "frozen-in field" have some interesting consequences.

For example, it is possible to move an ionized gas by altering magnetic fields, propelling it by magnetic pressure, and this forms the basis of ion drive. Also, it is possible to slow down or stop an ionized gas by running it into a region of increasing magnetic field or to use the interaction between a magnetic field produced in a body and a surrounding ionized gas to slow down the body, by magnetic braking. When a gas is so slowed

down the magnetic pressure, hence the magnetic field, is increased, thus causing increased currents to flow and enabling some of the kinetic energy of the gas to be taken off directly as electricity. By balancing gas pressure against magnetic pressure a mass of hot ionized gas may be isolated from its surroundings by a magnetic field, and it is this possibility which makes magnetohydrodynamics of such crucial importance in research on controlled thermonuclear reactions. The inverse of this arrangement, where a magnetic field is used to isolate a body surrounded by hot gas should be of great interest to the aeronautical scientist.

In addition to these steady processes, the dynamic processes in a magnetohydrodynamic fluid differ from those of a normal fluid. For instance, the equations (4.3) indicate that the velocity of sound in the fluid is anisotropic. In directions parallel to the magnetic field it remains unaltered as $c_{\parallel}$,

$$c_{\parallel}^2 = \frac{\partial p}{\partial \rho} = \frac{\gamma p}{\rho}, \qquad (5.1)$$

while perpendicular to the field it becomes $c_{\perp}$,

$$c_{\perp}^2 = \frac{\gamma p}{\rho} + \frac{B^2}{4\pi\rho}. \qquad (5.2)$$

In addition, because magnetic field lines behave as though elastic, transverse waves can propagate along the field lines, travelling with the Alfvén speed c_A

$$c_A^2 = \frac{B^2}{4\pi\rho}, \qquad (5.3)$$

and these transmitted waves violently alter the character of the velocity field, since vorticity is no longer conserved.

Since the velocity of sound perpendicular to the field increases with density, it is clear the compressive shock waves can form in a magnetohydromagnetic fluid, and by using the concepts of conservation of matter, momentum, energy, and magnetic flux, the magnetohydrodynamic analogues of the Rankine–Hugoniot equations can be written down.

$$[\rho u] = 0$$

$$[\rho u^2 + p + B^2/8\pi] = 0 \qquad (5.4)$$

$$\left[\tfrac{1}{2}u^2 + \frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{B^2}{4\pi\rho} \right] = 0.$$

6. THE PHYSICAL BASIS OF MAGNETOHYDRODYNAMICS

The conditions required for producing magnetohydrodynamic behaviour have been described, and it will be seen that for systems on a laboratory scale, magnetohydrodynamic behaviour is most easily produced in a diffuse hot plasma. This is the system of interest in controlled thermo-nuclear research, and I suspect that of interest in aerodynamics. It is also the system for which the connexion between the basic physical processes and the mathematical idealization is most obscure. The conceptual difficulties arise because of the long mean free paths of particles. For example, in an ionized gas at 1 atmosphere pressure at a temperature of 10^6 degrees the mean free path is ~ 4 metres, at 10^8 degrees it is $\sim 10^6$ metres, and it is not clear how a hydrodynamic treatment can be justified. I wish to show in what sense such a system behaves hydrodynamically, and to discuss the limitations of the description.

7. THE PARTICLE INTERACTION

An ionized gas is no continuous fluid, but is composed of many charged particles interacting through their electric fields, and it is our problem to discover in what sense magnetohydrodynamics describes the gross behaviour of such a system. Most plasmas are complicated by the presence of neutral atoms or incompletely stripped ions with which electrons can make complex inelastic collisions but if the temperature is high enough all ions are completely stripped and the gas consists solely of ions and electrons, which for most purposes make only elastic collisions. Even in this case the particle interaction is not simple, for the interaction falls off only as $1/r^2$, while the probability of finding a particle at distance r varies as r^2, hence there is no simple way of defining a collision distance r_0, such that two particles separated by a distance $r < r_0$ interact, while if the separation $r > r_0$ interaction is negligible. Instead we must think of each particle interacting with very many of its neighbours, as in a liquid, and it is necessary to consider the correlation between the motions of nearby particles. Fortunately, the forces acting on the particles are usually small, it is possible to evaluate these correlations by a number of techniques, and it is discovered that the principal effect of the small correlation between particle orbits is to introduce an effective screening.

As might be expected charges of like signs repel each other slightly, while those of opposite sign attract, and thus, on average, any charge is surrounded by a diffuse cloud of charges of the opposite sign, which serves to screen out its field in a distance λ_D,

$$\lambda_D^2 = \frac{kT}{4\pi \, ne^2}.$$

This result has been obtained in several ways, perhaps most elegantly

by noting that the plasma has a calculable dielectric constant $K(\omega,k)$ which depends on frequency and wave length of the applied field, thus the electric field due to a test charge is not just $\sim 1/r^2$, but a more elaborate function of the radius which finally vanishes as $1/r^2 \exp -r/\lambda_0$.

Furthermore, because of the slight bending of trajectories, the interaction can be treated linearly, and the effect of many-body encounters considered as the sum of many two-body encounters. Thus, in spite of many differences in the actual mechanism, the interaction between the charged particles in a plasma can be treated formally like the short range interaction between normal gas molecules, the true many-particle interaction being replaced by an effective two-body interaction—which however depends on temperature and density.

MAGNETIC ISOLATION.

MAGNETIC CONFINEMENT.
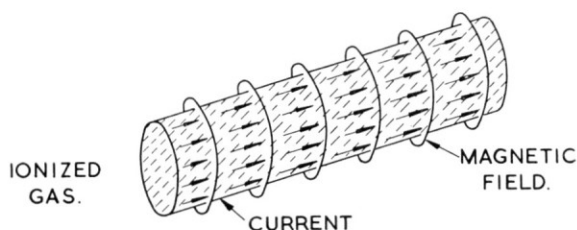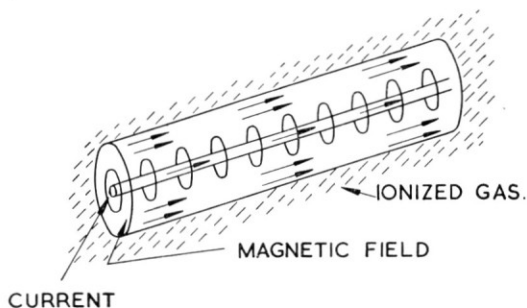
8. KINETIC THEORY AND THE DEDUCTION OF NORMAL HYDRODYNAMICS

In order to deduce hydrodynamics we may begin from the Boltzmann equation for the distribution function $f(\mathbf{x},\mathbf{v},t)$ for the assembly of particles, $f \, dx \, dy \, dz \, dv_x \, dv_y \, dv_z$ giving the probability of finding a particle at a position in the range $dx \, dy \, dz$ at $\mathbf{x}$ having a velocity in the range $dv_x \, dv_y \, dv_z$. at $\mathbf{v}$. Its time development occurs in two ways, first since particles change their positions and velocities due to their motion under the in-
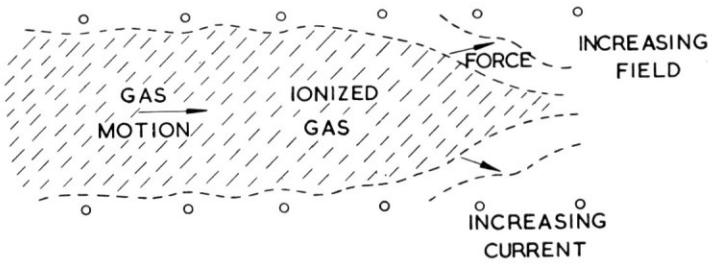
DIRECT CONVERSION.
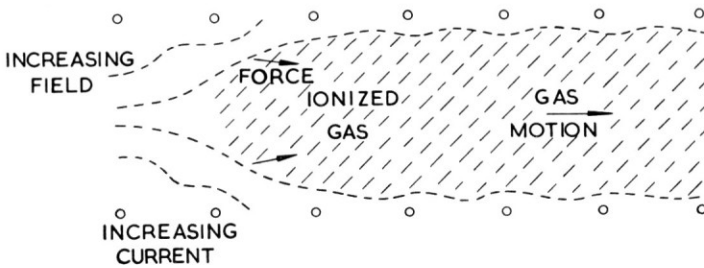
MAGNETIC PROPULSION.

fluence of given force fields **F**, and secondly because of discontinuous changes produced by collisions. The total change, the sum of these, can be shown to vanish from Liouville's theorem thus:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt} + \frac{\partial f}{\partial \mathbf{v}}\frac{d\mathbf{v}}{dt} + \frac{\partial f}{\partial t}\bigg|_{\text{coll}} \tag{8}$$

$$= \frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} + \frac{\mathbf{F}}{m}\cdot\frac{\partial f}{\partial \mathbf{v}} + \frac{\partial f}{\partial t}\bigg|_{\text{coll}} = 0. \tag{8.1}$$

The rate of change due to collisions represents the difference between the rate at which particles are knocked into the region **v** and the rate at which they are knocked out by collision with other particles, i.e.

$$\frac{\partial f}{\partial t}\bigg|_{\text{coll}} = -\int d\Omega \int d^3v'\,|\mathbf{v} - \mathbf{v}'|\,\sigma(\Omega)[f(\mathbf{v})f(\mathbf{v}') - f(\underline{\mathbf{v}})f(\underline{\mathbf{v}}')]. \tag{8.2}$$

By forming the moments of the B.E. with the mass, m, the momentum $m\mathbf{v}$ and the energy $\frac{1}{2}mv^2$, all being quantities unchanged by collision, one obtains three hydrodynamic equations:

$$\frac{\partial \rho}{\partial t} + \text{div} (\rho \mathbf{u}) = 0$$

$$\rho \frac{Du_i}{Dt} + \frac{\partial}{\partial x_j} p_{ij} - \mathscr{F}_i = 0 \tag{8.3}$$

$$\frac{3}{2} n \frac{D}{Dt} kT + p_{ij} \frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_i} q_i = 0$$

where

$$\rho = \int m f \mathrm{d}^3 v, \quad \rho \mathbf{u} = \int m \mathbf{v} f \mathrm{d}^3 v$$

$$p_{ij} = \int m(v - u)_i (v - u)_j f \mathrm{d}^3 v,$$

$$\frac{3}{2} kT = \int \frac{1}{2} m(v - u)^2 f \mathrm{d}^3 v \tag{8.4}$$

$$q_i = \int \frac{1}{2} m(v - u)^2 (v - u)_i f \mathrm{d}^3 v.$$

These equations have the form of hydrodynamic equations, but have no content until a method is discovered for determining p and q, and for displaying how these depend on local conditions. In order that they should, the distribution function must also depend on local conditions. If its form is determined almost entirely by collisions, it does so, and the problem can be solved. Formally this can be shown by introducing two times, a collision time τ_c defined by

$$\frac{1}{\tau_c} = \frac{1}{f(v)} \int \mathrm{d}\Omega \int \mathrm{d}^3 v' |\mathbf{v} - \mathbf{v}'| \sigma(\Omega) f(\mathbf{v}) f(\mathbf{v}') = n <\sigma v> \tag{8.5}$$

and a characteristic hydrodynamic time τ_h, determined by the nature of the motion. If $\tau_h >> \tau_c$, the Boltzmann equation may be solved by expanding in powers of τ_c/τ_h, and considering the approximate equations obtained by insisting that each coefficient vanish. This procedure leads, in the first approximation, to the equations of isotropic flow, and in the second, to the Navier-Stokes equation and the heat conduction equation for a compressible gas.

9. DEDUCTION OF MAGNETOHYDRODYNAMICS

For the ionized gas in a magnetic field one must use two distribution functions, but an exactly similar analysis can be given, the current density **j** being determined in addition.

In a magnetic field the method of analysis must be modified slightly and rather complex expressions are found for the transport coefficients, all of which become tensors (e.g. heat flow is not in the direction of the temperature gradient), whose interpretation requires some care.

If the magnetic field is very strong a difficulty of principle appears. In the term of the Boltzmann equation involving the acceleration,

$$\frac{e}{m} [\mathbf{E} + \mathbf{v} \times \mathbf{B}] \cdot \frac{\partial f}{\partial \mathbf{v}},$$

there appears a new characteristic frequency, the cyclotron frequency $\omega_L = eB/m$, which may often be much higher than the collision frequency $1/\tau_c$. In fact in just those hot ionized gases in strong fields where typical magnetohydrodynamic behaviour is to be expected the product $\omega_L \tau_c$ is particularly large; e.g. at a temperature of $10^{6\circ}$, pressure of $1/10$ atm and a field of 2000 G (conditions in ZETA) we find:

$$\omega_L = 7 \cdot 5 \times 10^5 \text{ sec}^{-1}, \ \tau_c = 8 \cdot 10^{-5} \text{ sec}, \ \omega\tau = 60, \text{ for ions, while}$$
$$\omega\tau = 6 \cdot 5 \times 10^3 \text{ for electrons}$$

and considerable doubt is cast on the entire expansion procedure. Indeed, for many magnetohydrodynamic processes, the collision frequency is less than the frequencies associated with macroscopic motion, and hydrodynamic behaviour is not produced by collisions. In fact, the collisions of particles often can be neglected altogether, the entire interaction occurring with the macroscopic field.

10. COLLISION-FREE MAGNETOHYDRODYNAMICS

If we consider the motion of charged particles in a magnetic field, an alternative method of arriving at magnetohydrodynamics suggests itself. In a uniform magnetic field the motion of particles parallel to the field is unaffected, but the component of the trajectory in the plane normal to the field lines is bent into a circle, and the particle rotates about the field lines with an angular frequency ω_L, in a circle of radius

$$r_L = \frac{v_\perp}{\omega_L}. \tag{10.1}$$

This restriction on particle motion localizes the trajectories to the magnetic field lines and introduces a medium like behaviour for motion across the field—the typical magnetohydrodynamic behaviour.

From what we have said already certain facts are apparent. Since there are no collisions, there is nothing to make the velocity distribution isotropic, however, the two components of velocity perpendicular to the field lines must be equal. This suggests that the pressure, while not a simple scalar, will contain two components $p_\parallel$ and $p_\perp$ which are independent.

If we study the motion of particles in magnetic fields, using the equation of motion

$$\frac{d\mathbf{v}}{dt} = \frac{e}{m}\left[\frac{\mathbf{F}}{e} + \mathbf{E} + \mathbf{v} \times \mathbf{H}\right] \tag{10.2}$$

we discover in addition to the rotating velocity, the drift velocity $\mathbf{v}_D$ across the field lines,

$$\mathbf{v}_D = \frac{\left(\dfrac{\mathbf{F}}{e} + \mathbf{E}\right)}{H^2} \times \mathbf{H}. \tag{10.3}$$

If we consider the drift of two particles of opposite sign, we get the mean velocity $\mathbf{u}$ by adding

$$\mathbf{u} = \frac{\mathbf{E} \times \mathbf{H}}{H^2}, \tag{10.4}$$

and the relative velocity $\mathbf{w}$, related to the current by subtracting,

$$\mathbf{w} = \frac{1}{e} \frac{\mathbf{F} \times \mathbf{H}}{H^2}. \tag{10.5}$$

Thus, paradoxically, an electric field produces no current, but a drift of the matter, whereas any non-electric force produces a current. This result must be used with caution, since often the boundary conditions do not permit current to flow, and the immediate effect of the force $\mathbf{F}$ is to produce a charge separation and an electric field perpendicular to $\mathbf{F}$, whereupon the electric field produces a velocity $\mathbf{u}$ in the direction of $\mathbf{F}$—a process which often makes a magnetically confined plasma unstable. An important force $\mathbf{F}$ is that associated with the centrifugal acceleration of particles moving along curved field lines, which gives rise to a velocity

$$v_D = \frac{1}{R} \frac{m v_\parallel^2}{eH} = \frac{r_L}{R} v_\parallel$$

where R is the radius of curvature of the field lines.

The consequences of such particle motions can be obtained formally by solving the collision-free Boltzmann equation by an expansion procedure which utilizes the smallness of the Larmor radius r_L in comparison with the distance over which the fields change. In zero order, we obtain the drift velocity $\mathbf{E} \times \mathbf{H}/H^2$, and the statement that the distribution function f is a function of the two velocity components $v_\parallel$, $v_\perp$, hence that the pressure tensor has the two expected components $p_\parallel$, $p_\perp$. In first order we obtain the currents due to effects described above, and in addition a current due to density gradients. In steady state these are just those needed to satisfy the magnetohydrostatic equation

$$\nabla p = \mathbf{j} \times \mathbf{H}.$$

As an example consider a diffuse plasma with a density gradient in the OX direction confined in a magnetic field in the OZ direction. The current that flows is a consequence of the imperfect cancelling of the circulating velocities of individual particles

$$= n_+ \, e v_+ - n_- \, e v_- = j e \left\{ \int\!\!\int v_\perp \sin\theta f^+ (v,x + r_L \sin\theta) - \right.$$
$$\left. - f^- (v,x + r_L \sin\theta) \right.$$

$$= e \int_0^{2\pi} d\theta \int dv \, v_\perp \sin\theta \left[\frac{\partial f^+}{\partial x} r_L^+ \sin\theta - \frac{\partial f^-}{\partial x} r_L^- \sin\theta \right] \qquad (10.6)$$

$$= \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} d^3 v \frac{v_\perp^2}{B} \left[m_+ \frac{\partial f^+}{\partial x} + m_- \frac{\partial f^-}{\partial x} \right] \sin^2\theta$$

$$= \frac{1}{B} \frac{\partial}{\partial x} (p_+ + p_-), \text{ if } B \text{ is effectively constant.}$$

Hence

$$\dot{j}_y \, B_z = \frac{\partial p}{\partial x}.$$

By similar or more formal arguments the magnetohydrostatic results can be obtained.

To discuss the possible motions of a collision free magnetohydrodynamic system, one notes first that the relation between zero order drift velocity and electric field $\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$ is exactly that for a perfect conductor, thus flux trapping occurs and the field is frozen to the fluid. If by external forces the fluid is forced to move along the field lines, the magnetic field has no effect, but the components of velocity $v_\parallel$ and $v_\perp$ vary independently, thus the perpendicular component of velocity is unaltered, and the pressure increases solely because of the increase in density, $p_\perp \sim \rho$. On the other hand, only a single degree of freedom is associated with the parallel velocity component and this increases $p_\parallel \sim \rho^3$. If the gas is slowly compressed perpendicular to the field lines, the field is slowly increased and $v_\perp$ is increased by betatron acceleration in such a way that the flux threading the Larmor orbits is unchanged, i.e.

$$\frac{v_\perp^2}{B} = \text{const.},$$

and since $B \sim \rho$

$$p_\perp \sim \langle \rho \, v_\perp^2 \rangle \sim \rho^2,$$

i.e. the perpendicular component of pressure behaves like that of a gas with two degrees of freedom. At the same time $p_\parallel \sim \rho$.

For dynamic problems one can again solve the Boltzmann equation by a perturbation procedure, and for motion strictly perpendicular to the field lines these adiabatic results apply. However, if strict symmetry along field lines is not preserved the perturbed distribution function, which can be expressed in terms of the perturbed fields and the unperturbed function, is expressed by an integral along field lines, and as might be expected no true local hydrodynamics is obtained—nothing localizes flow along field lines.

Even the two-dimensional hydrodynamics so obtained is strictly linearized, but some non-linear analogues are easily seen. Here, however, a difficulty arises. It is known that in normal hydrodynamics the structure of a shock wave can be obtained by using the dissipative terms in the Navier–Stokes equation: that is by using the first order terms in an expansion in the mean free time, or mean free path, and it might be imagined that in a similar way a magnetohydrodynamic shock might be obtained by expanding in powers of the Larmor radius. This is not the case, because the adiabatic relation (10.7) has been shown to hold to all orders and the result $B \sim \rho$ is one of the shock conditions, thus $p_\perp \sim \rho^2$, and only adiabatic flow normal to the field lines can be obtained.

Summary—We have seen that the most interesting magnetohydrodynamic phenomena occur in plasmas and that, on the laboratory scale, the mean free path is often long. However, we have shown how strictly two-dimensional linearized magnetohydrodynamics can be obtained from the motion of non-colliding particles. The derivation is not complete, since it is not capable of producing local hydrodynamic equations for non-symmetric flow, or even in the symmetric case for non-adiabatic flow.

REFERENCES

Section 1   S. CHAPMAN and V. C. A. FERRARO, *Nature, Lond.* Vol. 126, p. 129, 1930.
                   T. G. COWLING, *MNRAS.* Vol. 94, p. 39, 1933.
                   J. HARTMANN and F. LAZARUS, *K. Danske. Vidensk. Selsk.* Vol. 15 (5 and 6), 1937.

Sections 2, 3, 4, 5   H. ALFVÉN, *Cosmical Electrodynamics*, Oxford, 1953.
                   T. G. COWLING, *Magnetohydrodynamics*, New York, 1957.
                   F. DE HOFFMAN and E. TELLER, *Phys. Rev.* Vol. 80, p. 692, 1950.

Section 7   D. PINES and D. BOHM, *Phys. Rev.* Vol. 85, p. 338, 1952.
Section 8   D. HILBERT, *Math. Ann.* Vol. 71, p. 562, 1912.
                   S. CHAPMAN and T. G. COWLING, *Mathematical Theory of Non-uniform Gases*, Cambridge, 1939.

Section 9   W. MARSHALL, AERE T/R.2419, 1958.
                   R. LANDSHOFF, *Phys. Rev.* Vol. 76, p. 904, 1949.

Section 10   L. SPITZER, *Physics of Fully Ionized Gases*, Interscience, New York, 1956.
                   K. M. WATSON, *Phys. Rev.* Vol. 102, p. 12, 1956.
                   K. M. WATSON and K. A. BRUECKNER, *Phys. Rev.*, Vol. 102, p. 19, 1956.
                   G. F. CHEW, M. L. GOLDBERGER and F. E. LOW, *Proc. Roy. Soc.* A Vol. 236, p. 112, 1956.
                   M. D. KRUSKAL, *Nuovo Cimento* (in publication).